Int J Theor Phys
DOI 10.1007/s10773-011-1000-y

Fractional Pais–Uhlenbeck Oscillator

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Received: 17 July 2011 / Accepted: 20 October 2011

Abstract In this paper we study the fractional Lagrangian of Pais–Uhlenbeck oscillator. We obtained the fractional Euler–Lagrangian equation of the system and then we studied the obtained Euler–Lagrangian equation numerically. The numerical study is based on the so-called Grünwald–Letnikov approach, which is power series expansion of the generating function (backward and forward difference) and it can be easy derived from the Grünwald–Letnikov definition of the fractional derivative. This approach is based on the fact, that Riemman–Liouville fractional derivative is equivalent to the Grünwald–Letnikov derivative for a wide class of the functions.

Keywords Riemann–Liouville derivatives · Pais–Uhlenbeck oscillator · Grünwald–Letnikov approach

1 Introduction

As it is known the Pais–Uhlenbeck oscillator is used to describe a higher derivative theory [1]. In field theories higher derivative were introduced in order to get rid of ultravio-

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Published online: 05 November 2011

let divergences [2]. It is thought that higher-order theories would possess propagators having poles with non-positive residues (i.e. non-positive norms) and would therefore threaten the unitarity of the theory. But an interesting solution to such difficulty has been recently proposed in [3] by finding a quantum transformation which gives its Hamiltonian a non-Hermitian, *PT* symmetric form.

During the last decades the fractional calculus that is an extension of classical calculus was subjected to an intense debate in various fields of science and engineering [4–10]. Many authors have paid a considerable attention to the formulation of the fractional Euler–Lagrange problem. Riewe investigated non-conservative Lagrangian and Hamiltonian mechanics and for those cases formulated a version of the Euler–Lagrange equations [11]. Further works on Lagrangian and Hamiltonian approaches can be found in Refs. [12–16] and the references therein. Since the fractional Euler–Lagrange equations contain both the left and the right fractional derivatives the analysis of them is a new and interesting subject for both mathematical and physical point of view.

This type of equations is obtained when the minimum action principle and fractional integration by parts rule are applied. In our knowledge very few results were reported on this type of new fractional equations.

Numerical analysis of fractional differential equations appeared in many researches [17–20]. For example, recently, Podlubny [21], and Podlubny et al. [22] introduces how to numerically solve differential equations using matrix form representation.

This manuscript is focused on fractional Euler–Lagrange equation of the Pais–Uhlenbeck oscillator.

This paper is organized as follows:

In Sect. 2, the basic definitions of fractional derivatives are discussed briefly. In Sect. 3, we study the fractional Pais–Uhlenbeck model. In Sect. 4, numerical analysis of the corresponding fractional Euler–Lagrange equation is carried out. The paper closes with concluding remarks.

2 Basic Tools

In this section, we briefly review the definitions of the fractional derivatives.

These definitions are used in the Lagrangian formulation and the solution of examples leading to the equation of motion of the fractional order.

The left Riemann–Liouville fractional integral is defined as follows [4, 5]

$$ {}_aI_t^\alpha x(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-\tau)^{\alpha-1} x(\tau) d\tau. \tag{1} $$

The right Riemann–Liouville fractional integral has the form

$$ {}_tI_b^\alpha x(t) = \frac{1}{\Gamma(\alpha)} \int_t^b (\tau-t)^{\alpha-1} x(\tau) d\tau. \tag{2} $$

The left Riemann–Liouville fractional derivative reads as

$$ {}_aD_x^\alpha f(x) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dx} \int_a^x \frac{f(\tau)}{(x-\tau)^\alpha} d\tau. \tag{3} $$

The right Riemann–Liouville fractional derivative is given by

$$ {}_xD_b^\alpha f(x) = -\frac{1}{\Gamma(1-\alpha)} \frac{d}{dx} \int_x^b \frac{f(\tau)}{(\tau-x)^\alpha} d\tau. \tag{4} $$

Here α is the order of the derivative such that $0 < \alpha \leq 1$, Γ denotes the Euler's Gamma function. When α becomes an integer, these derivatives become

$$ {}_aD_x^\alpha f(x) = \left(\frac{d}{dx}\right)^\alpha f(x), $$
$$ {}_xD_b^\alpha f(x) = \left(-\frac{d}{dx}\right)^\alpha f(x); \quad \alpha = 1, 2, \ldots, \tag{5} $$

namely, we recover the usual classical derivatives.

3 The Model

The model of the Pais–Uhlenbeck oscillator consists of a one dimensional harmonic oscillator Lagrangian plus a term quadratic in acceleration [1].

In a series of recent papers the Pais–Uhlenbeck model has been introduced and studied in many details [23–26]. The model is interesting by itself and in connection with gravity, since it involves a differential equation of order higher than two. More explicitly, this model is characterized by the following fourth-order differential equation:

$$ \frac{d^4x}{dt^4} + (w_1 + w_2)\frac{d^2x}{dt^2} + w_1^2 w_2^2 x = 0, \tag{6} $$

where w_1 and w_2 are real numbers. The quantization of this model, if not properly performed, gives rise to some peculiarities like ghosts (i.e. negative norm states) and a Hamiltonian which is not bounded from below. In [23] and references therein the authors propose a strategy to quantize the model which cures these problems, at least if $w_1 \neq w_2$. A different quantization strategy is proposed, which produces a self-adjoint Hamiltonian which is bounded from below was suggested in [24]. In order to see other possibilities we recommend Refs. [25, 26].

It was shown that a complex canonical transformation takes the fourth order derivative Pais–Uhlenbeck oscillator into two independent harmonic oscillators, which means that this model has energy bounded from below, unitary time-evolution and no negative norm states, or ghosts.

For these reasons the model is interesting to be analyzed.

In this section we will study the so-called Pais–Uhlenbeck oscillator. Its classical Lagrangian is given as

$$ L_{PU} = \frac{1}{2}\ddot{x}^2 - \frac{(w_1^2 + w_2^2)}{2}\dot{x}^2 + \frac{w_1^2 w_2^2}{2}x^2. \tag{7} $$

The above Lagrangian can be generalized and written in fractional form as:

$$ L_{PU}^F = \frac{1}{2}({}_aD_t^{2\alpha}x)^2 - \frac{w_1^2 + w_2^2}{2}[{}_aD_t^\alpha x]^2 + \frac{w_1^2 w_2^2}{2}x^2 \tag{8} $$

for an appropriate space of functions [4].

By using the fractional integration by parts the fractional Euler–Lagrange equation can be obtained as follows

$$ \frac{\partial L}{\partial x} + {}_tD_b^\alpha \frac{\partial L}{\partial_a D_t^\alpha x} + {}_tD_b^{2\alpha}\frac{\partial L}{\partial_a D_t^{2\alpha} x} = 0. \tag{9} $$

Now, making use of (8) we can rewrite (9)

$$ (w_1^2 w_2^2)x - (w_1^2 + w_2^2){}_tD_b^\alpha {}_aD_t^\alpha x + {}_tD_b^{2\alpha} {}_aD_t^{2\alpha} x = 0. \tag{10} $$

The above equation is a fractional differential equation containing a composition of left and right fractional derivatives. Now, our aim is to obtain a numerical solution for (10). We notice that when $\alpha \to 1$, we obtained the classical Euler–Lagrange equation from (1).

4 Numerical Results of Fractional Euler–Lagrange Equation

For numerical solution of the linear fractional-order equation (10) we can use the decomposition to its canonical form with substitution $x \equiv x_1$. We obtain the set of equation in the form:

$$\begin{aligned} {}_aD_t^\alpha x_1 &= x_2 \\ {}_tD_b^\alpha x_2 &= x_3 \\ {}_aD_t^\alpha x_3 &= x_4 \\ {}_tD_b^\alpha x_4 &= -(w_1^2 w_2^2)x_1 + (w_1^2 + w_2^2)x_3 \end{aligned} \tag{11}$$

where we can set four initial conditions $x_1(0), x_2(0), x_3(0), x_4(0)$. Instead left and right side Riemann–Liouville fractional derivatives (3) and (4) in the set of (11) can be used the left and right Grünwald–Letnikov derivatives, which are equivalent to the Riemann–Liouville fractional derivatives for a wide class of the functions [5]. The Grünwald–Letnikov derivatives can be defined by using upper and lower triangular strip matrices (Podlubny's matrix approach) or we can directly apply the formula derived from the Grünwald-Letnikov definitions, backward and forward, respectively, for discrete time step kh, $k = 1, 2, 3, \ldots$. Le us consider the second approach, which works very well for linear as well as for nonlinear fractional differential equations [27]. Then, general numerical solution of the fractional linear differential equation with left side derivative in the form

$${}_aD_t^\alpha x(t) = f(x(t), t) \tag{12}$$

can be expressed for discrete time $t_k = kh$ in the following form:

$$x(t_k) = f(x(t_k), t_k)h^\alpha - \sum_{i=m}^{k} c_i x(t_{k-i}) \tag{13}$$

where $m = 0$ if we do not use a short memory principle, otherwise it can be related to memory length. The binomial coefficients c_i, $i = 1, 2, 3, \ldots$, can be calculated according to relation

$$c_i = \left(1 - \frac{1+\alpha}{i}\right) c_{i-1} \tag{14}$$

for $c_0 = 1$. Similarly we can derive a solution for an equation with right side fractional derivative.

In Fig. 1(a)–(f) are depicted the simulation results of (10) for various parameters w_1, w_2, and order α, where $a = b = 0$, for total simulation time 5 s and computational step $h = 0$.

5 Conclusions

Fractional Euler–Lagrange equations are new kind of fractional differential equations which combine the left and the right fractional derivative. These kind of fractional differential equations are of a new type and finding the numerical solutions of them is an intriguing issue. Particularly, when we fractionalize a Lagrangian corresponding to a theory having

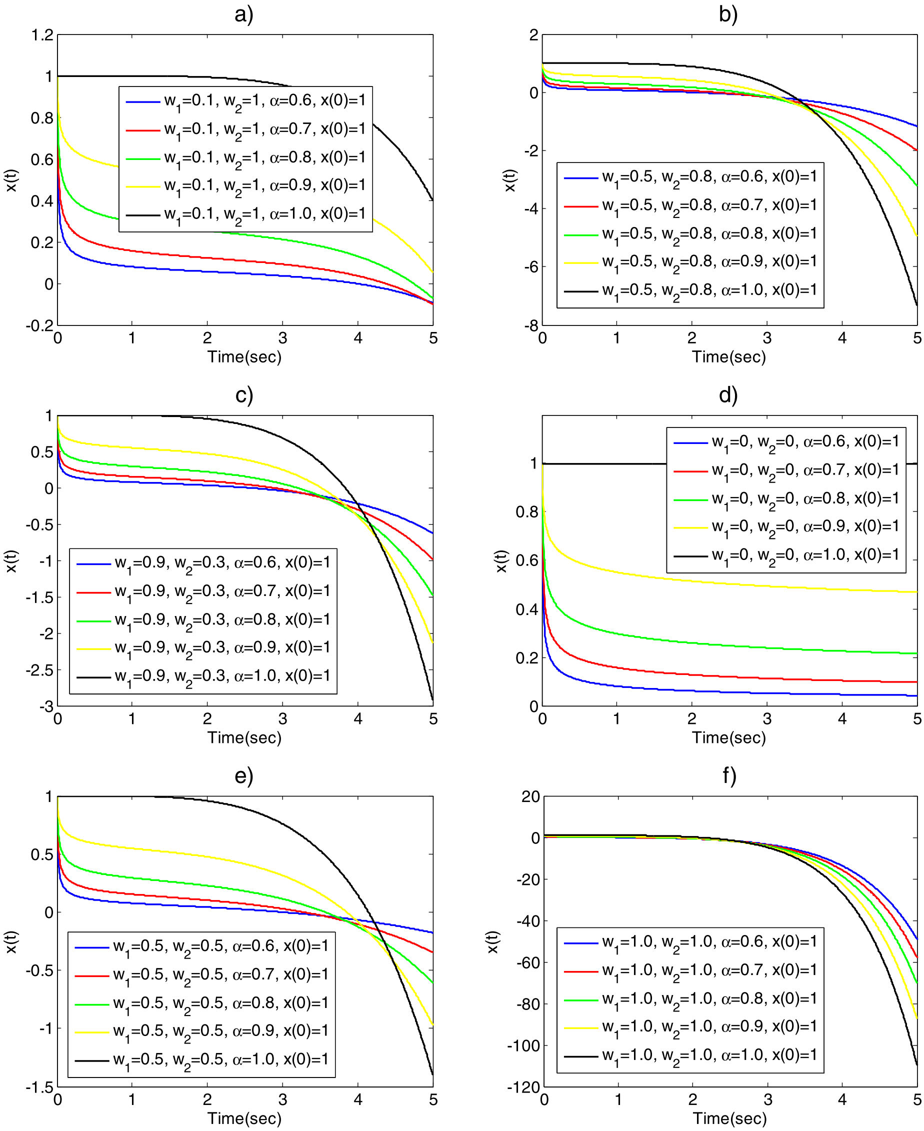

Fig. 1 Simulation results for various parameters and fractional derivative orders

higher order derivatives a natural question is what is the numerical behavior of the obtained Euler–Lagrange equations. On this line of taught in this paper we investigated the numerical solutions of the fractional Pais–Uhlenbeck Euler–Lagrange equation. For this reason we started with a classical Lagrangian and then we fractionalized it and we obtained the fractional Euler–Lagrange equation. After that we investigated numerically the solution of the fractional Euler–Lagrange equation. The numerical results depicted in Fig. 1 show clearly that for various values of the parameters w_1 and w_2 the behaviors of the fractional Euler–Lagrange equation strongly depend on the order of the fractional derivative. For each graph we provided the classical solution of the equations and five different cases for the value of the parameters w_1, w_2 and α. For example, in Fig. 1d the classical solution is constant but the presence of both left and right derivatives makes the solution of the fractional having decaying behaviors.

Acknowledgements The authors would like to that to the referee for his (her) very useful comments and remarks.

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